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Multilevel Methods for the Simulation of Turbulence

A Simple Model

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Our aim in this article is to present for a very simple model—namely a pair of ordinary coupled differential equations—some of the features of the multilevel numerical methods which have been introduced recently for the numerical simulation of turbulent flows. The two components of this simple differential system are intended to represent the large and small scale components of a flow. We show that many new numerical schemes can be introduced by treating differently the small and large scale components; also different time steps can be used for these two components. The stability analysis which we conduct for this simple model shows that these new multilevel schemes can produce a substantial saving in computing time, although the stability analysis leads sometime to counterintuitive conclusions. The error analysis for this model will be conducted elsewhere. Also the reader is referred to the articles quoted below (in particular, [8, 10]) for the utilization of similar multilevel schemes for the Navier–Stokes equations themselves.

1. INTRODUCTION

Some numerical algorithms have been introduced in a recent past for the numerical simulation of turbulent flows, namely the nonlinear Galerkin method and the incremental unknowns method (see, e.g., [3–16, 24–26, 30–34]). These methods initially motivated by some theoretical results in dynamical systems theory are now developing into more conventional numerical methods for which the essential feature and novelty is a differentiated treatment of the high and low frequency components of a flow or its small and large structures. It is indeed a recognized fact of the dynamical system theory approach (attractors, exact and approximate inertial manifolds), that the high and low frequency components of a flow play different roles (see, among many Refs., [11–16]). This fact is also well recognized in conventional turbulence theory (see, e.g., Batchelor's book [1, Chap. VI]) that the high frequencies, commanded in fact by a high viscosity, attain a statistical equilibrium much faster than the low frequencies.

The simple differential system that we propose here reads

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1/\varepsilon \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} yz \\ -y^2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \quad t > 0, \qquad (1)$$

$$y(0) = y_0, \quad z(0) = z_0. \qquad (2)$$

Here y and z play the same role respectively as the low and high frequency components of the flow (e.g., of the velocity vector). The important features of the matrix in (1) is that its eigenvalues are $O(1)$ and $O(1/\varepsilon)$, $\varepsilon > 0$ small; the linear coupling terms ($y - z$ and $z - y$ terms) are not important and in fact are not present in the related case of Fourier and spectral discretizations.

For this very simple system and also for the associated linear system obtained by dropping the nonlinear terms, the stability analysis is elementary. We will show how one can build appropriate numerical schemes based on different treatments and different time steps for y and z ($\Delta t_y \neq \Delta t_z$). Although it may not be transparent at first glance, much of the analysis is very similar to what has been done elsewhere in infinite dimensions for partial differential equations and, in particular, the Navier–Stokes equations (see, e.g., [3, 25, 31]). We recover many of the results of the stability analysis of the multilevel methods for the Navier–Stokes equations without paying the price of all the necessary mathematical machinery related, e.g., to the discrete Sobolev space norms. *As we will see the conclusions of the stability analysis are sometime counterintuitive.*

The utilization of the new multilevel methods are also justified by some other aspects which are not alluded to here: error analysis, count of the number of operations, and finally, the filtering role of the utilization of approximate inertial manifolds. For the error analysis and the count of operations see, in particular and for the Navier–Stokes equations [3, 22, 24]; for the present model see [19]; for the filtering role of approximate inertial manifolds, see, e.g., [8–10]. Also our equations used to model the stability analysis of the multilevel methods and the general behaviour of the solutions are probably too simple to model more involved physical phenomena. With that respect they

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depart from, e.g., the turbulent cascade models such as the GOY model [2, 18, 20, 21, 23, 27, 29, 35].

In Sections 1 and 2 we describe some numerical schemes and their stability properties. Then in Section 3 we briefly mention some numerical computations, adding, in particular, a white noise to make the dynamics more interesting. In the concluding remarks we allude to some other less elementary aspects of the nonlinear Galerkin method and the incremental unknowns method in the way they are implemented at this time (see, e.g., [34] and the references therein, or [4, 9, 10]).

We conclude this introduction with a few simple remarks on the system (1). For simplicity f and g are constants independent of time and of ε, i.e., $f, g = O(1)$. We use the analog of the energy equation in fluid mechanics, obtained by multiplying the first equation (1) by y, the second by z and adding the resulting equations.[1] From this we find that for any y_0, z_0, $y(t)$ and $z(t)$ are respectively of order $O(1)$ and $O(\varepsilon)$ for large time and, hence, $\dot{y} = O(1)$, $\dot{z} = O(1)$, so that $y/\dot{y} = O(1)$, $z/\dot{z} = O(\varepsilon)$. If $y_0 = O(1)$ and $z_0 = O(\varepsilon)$, the same result is valid for all times. Here large time play the role of times for which the statistical equilibrium has been reached and small time plays the role of transient time in turbulence; $y/\dot{y}$ is the analog of the large eddies turnover time, $z/\dot{z}$ is the analog of the small eddies turnover time.[2]

Another feature of Eq. (1) is the following: at first glance, because of the factor $1/\varepsilon$, Eq. (1) seems to be a stiff differential equation. In fact the factor $1/\varepsilon$, corresponding to an increased effective viscosity, has a beneficial damping effect and the stiffness of (1) may only appear in the initial transient if z_0 is not already small, $O(\varepsilon)$. This initial transient time is of little importance since we are interested in large time behavior and time averages.

2. TWO LEVEL DISCRETIZATION SCHEMES ($\Delta t_y = \Delta t_z$)

As indicated before, the schemes that we consider correspond to differentiated treatment for y and z; by this we mean that the time discretization (explicit/implicit) is not the same for y and z. By comparison with a partial differential equation for the unknown u, the introduction of the decomposition $u = y + z$ yields a number of new schemes by combining explicit/implicit discretizations for y and z, for the linear and nonlinear terms.

To bring a little more generality in the calculations, we replace the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1/\varepsilon \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \nu & \nu \\ \nu & \nu/\varepsilon \end{pmatrix}$$

and we start by considering the linear equation. An ordinary explicit Euler scheme with time step Δt, reads

$$\begin{aligned} &\frac{y^n - y^{n-1}}{\Delta t} + \nu y^{n-1} + \nu z^{n-1} = f, \\ &\frac{z^n - z^{n-1}}{\Delta t} + \nu y^{n-1} + \frac{\nu}{\varepsilon} z^{n-1} = g. \end{aligned} \tag{3}$$

It is standard that the stability (CFL-type) condition for (3) reads

$$\Delta t < 2\varepsilon/\nu, \tag{4}$$

so that for, e.g., $\nu = O(1)$ and ε small, Δt must be less than $O(\varepsilon)$.

If we treat y and z differently, we could consider the scheme

$$\begin{aligned} &\frac{y^n - y^{n-1}}{\Delta t} + \nu y^{n-1} + \nu z^{n-1} = f, \\ &\frac{z^n - z^{n-1}}{\Delta t} + \nu y^n + \frac{\nu}{\varepsilon} z^n = g. \end{aligned} \tag{5}$$

It is elementary to derive the stability condition for (5); we find

$$\Delta t < \frac{2 - 4\varepsilon}{\nu} \tag{6}$$

and

$$\Delta t < \frac{\nu}{\varepsilon}. \tag{7}$$

Similarly for the full nonlinear equation, we could consider the scheme

$$\begin{aligned} &\frac{y^n - y^{n-1}}{\Delta t} + \nu y^{n-1} + \nu z^{n-1} + y^{n-1} z^{n-1} = f, \\ &\frac{z^n - z^{n-1}}{\Delta t} + \nu y^n + \frac{\nu}{\varepsilon} z^n - (y^n)^2 = g. \end{aligned} \tag{8}$$

This scheme is linearly implicit in z and fully explicit in y (y^n is known when we compute z^n in the second equation (8)).

We can conduct the stability analysis of this scheme. We properly combine the three equations obtained as follows: we multiply the first equation (8) by y^{n-1}, the second equa-

[1] The stability analysis as well will be based on energy methods, i.e., on the discrete analog of the energy equation.

[2] It is a well-known fact in conventional turbulence theory that the eddy turn over time of small eddies (high frequencies) is much smaller than that of large eddies; see, e.g., [1; 28, p. 279].

tion (8) by z^n and the first equation (8) by $y^n - y^{n-1}$. After some lengthy but straightforward calculations we find that for any $\delta > 0$,

$$\begin{aligned}(|y^{n+1}|^2 + |z^n|^2) - (|y^n|^2 + |z^{n-1}|^2) + |z^n - z^{n-1}|^2 \\ + 2\kappa_1 \nu \Delta t\, |y^n|^2 + \kappa_2 \frac{\nu \Delta t}{\varepsilon} |z^n|^2 \\ \le 2\, \Delta t\, f y^n + (\Delta t)^2 \left(1 + \frac{3}{\delta}\right) (f)^2 \\ - 2\, \Delta t\, g z^n + \Delta t^2 |y^n z^n|^2,\end{aligned} \tag{9}$$

with

$$2\kappa_1 = 2 - 4\varepsilon - \nu \Delta t(1 + \delta),$$
$$\kappa_2 = 1 - \varepsilon \nu \Delta t(1 + 3/\delta).$$

Hence, if (6) holds, there exists $\delta > 0$ such that $2\kappa_1 > 0$. We then require

$$\Delta t < \frac{1}{2\varepsilon \nu (1 + 3/\delta)}, \tag{10}$$

which implies $\kappa_2 \ge \frac{1}{2}$.

Returning to (9) we find

$$\begin{aligned}\xi^{n+1} - \xi^n + |z^n - z^{n-1}|^2 + \kappa_1 \nu \Delta t |y^n|^2 \\ + \frac{\nu \Delta t}{\varepsilon} \left(\frac{1}{4} - \varepsilon L_n^2 \frac{\Delta t}{\nu} \gamma^2\right) |z^n|^2 \le \alpha\, \Delta t,\end{aligned} \tag{11}$$

with

$$\xi^n = |y^n|^2 + |z^{n-1}|^2,$$
$$\alpha = \frac{1}{\kappa_1} |f|^2 + \Delta t^2 \left(1 + \frac{3}{\delta}\right) |f|^2 + 4 \frac{\varepsilon \Delta t}{\nu} |g|^2,$$

and $L_n = \xi^1 + \alpha n\, \Delta t$. Hence, with $N = T/\Delta t$, $T > 0$ fixed, we conclude that

$$|y^n|^2 + |z^{n-1}|^2 \le L_1 + \alpha T \quad \text{for } n = 1, ..., N, \tag{12}$$

provided (6) and (10) are supplemented by

$$\Delta t \le \frac{\nu}{8\varepsilon (L_1 + \alpha T)}. \tag{13}$$

In conclusion, for ε small, for both the linear and nonlinear equations, the stability condition is that of the y equation (namely (6)). Hence, by considering a scheme which is explicit in y and implicit in z, we can take a mesh Δt commanded by the stability condition of the large eddies, which is much larger than what is allowed by (4).

In the PDE (partial differential equation) analog, one could argue that the resolution of the second equation (8) will be costly but this is not the case because z is small after the transient, and therefore many less digits/much less accuracy are needed when solving the second equation (8). Also this difficulty of solving the z equation is totally removed in the schemes considered in Section 2, where $\Delta t_y \neq \Delta t_z$.

3. MULTILEVEL DISCRETIZATION SCHEMES ($\Delta t_y \neq \Delta t_z$)

Pursuing the idea of treating differently y and z ("the large and small structures"), we consider now schemes with $\Delta t_y \neq \Delta t_z$; see [3, 24] for related schemes.

We set $\Delta t_y = \Delta t$, $\Delta t_z = q\, \Delta t$, where $q > 1$ is an integer. One of the many schemes one can think of is

$$\begin{aligned}\frac{y^{n+1/q} - y^n}{\Delta t} + \nu y^n + \nu z^n + \gamma y^n z^n = f, \\ \frac{y^{n+2/q} - y^{n+1/q}}{\Delta t} + \nu y^{n+1/q} + \nu z^n + \gamma y^n z^n = f, \\ \vdots \\ \frac{y^{n+1} - y^{n+(q-1)/q}}{\Delta t} + \nu y^{n+(q-1)/q} + \nu z^n + \gamma y^n z^n = f,\end{aligned} \tag{14}$$

$$\frac{z^{n+1} - z^n}{q\, \Delta t} + \nu y^{n+1} + \nu \frac{z^{n+1}}{\varepsilon} - \frac{\gamma}{q} \sum_{j=1}^{q} |y^{n+(j-1)/q}|^2 = g. \tag{15}$$

We take $\gamma = 0$ in the linear case and $\gamma = 1$ for the nonlinear equation. Note that the scheme is always explicit in y (y^{n+1} is known when computing z^{n+1} in (15)); it is linearly implicit in z (in (15)).

In the linear case, the stability analysis conducted as before leads to the stability conditions,

$$\Delta t < \frac{2 - 2\varepsilon - 2q\varepsilon}{\nu}, \tag{16}$$

$$\Delta t < \frac{1 - \varepsilon}{2\varepsilon \nu (1 + 2/\delta)}, \tag{17}$$

for some suitable $\delta > 0$. Of course, (17) follows from (16) for ε small and we conclude that the (main) stability condition (16) is very close, essentially the same as the stability condition for the y equation only (i.e., (4) when $z \equiv 0$). In the nonlinear case the stability analysis is more involved than in the two-level case. It leads, however, to two stability conditions, one very similar to (16), i.e.,

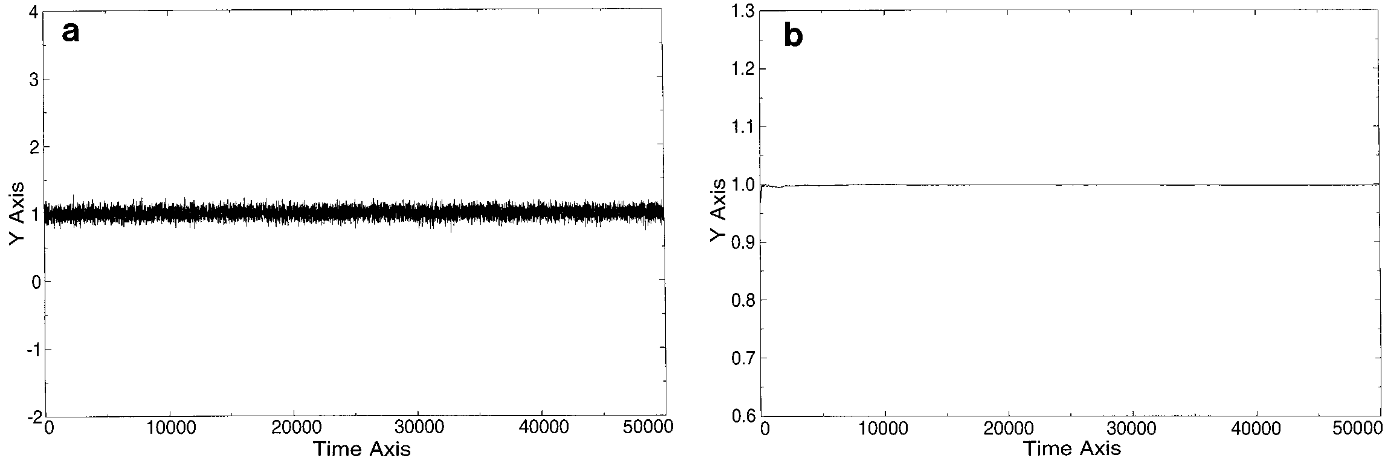

FIG. 1. (a) Instantaneous and (b) time-averaged values of $y(t)$.

$$\Delta t \leq \frac{2 - 4\varepsilon}{\nu + (\varepsilon L)^{1/2}}, \tag{18}$$

where L is some constant and another condition similar to (17), and which, like (17), follows from (18) for ε small. Hence for ε small the stability condition is essentially that of y and it is much better than (4).

4. SOME NUMERICAL COMPUTATIONS

Some numerical simulations have been performed which are, of course, very easy. The purpose is not to check the stability conditions or efficiency of the schemes but just to display the geometry of the solutions. To produce an interesting dynamics and for a better similarity with fluid mechanics, a white noise has been added. Hence, we solve

$$\dot{y} + y + z + yz = f + \sigma,$$

$$\dot{z} + y + \frac{1}{\varepsilon} z - y^2 = g + \tau,$$

$$y(0) = y_0, \quad z(0) = z_0,$$

where σ and τ are independent white noises. In fact, after time discretization with time step Δt we write

$$\int_s^{s+\Delta t} d\sigma(t) = \sigma(s + \Delta t) - \sigma(s) \sim \mathcal{N}(0, r\,\Delta t),$$

where $\mathcal{N}(0, \Delta t)$ is a random variable with zero mean and variance $r\,\Delta t$.

The results (Figs. 1 to 6) are displayed for $\varepsilon = 10^{-3}$, $f = g = 1$, $y_0 = 1$, $z_0 = 0.1$, $r = 0.01$.

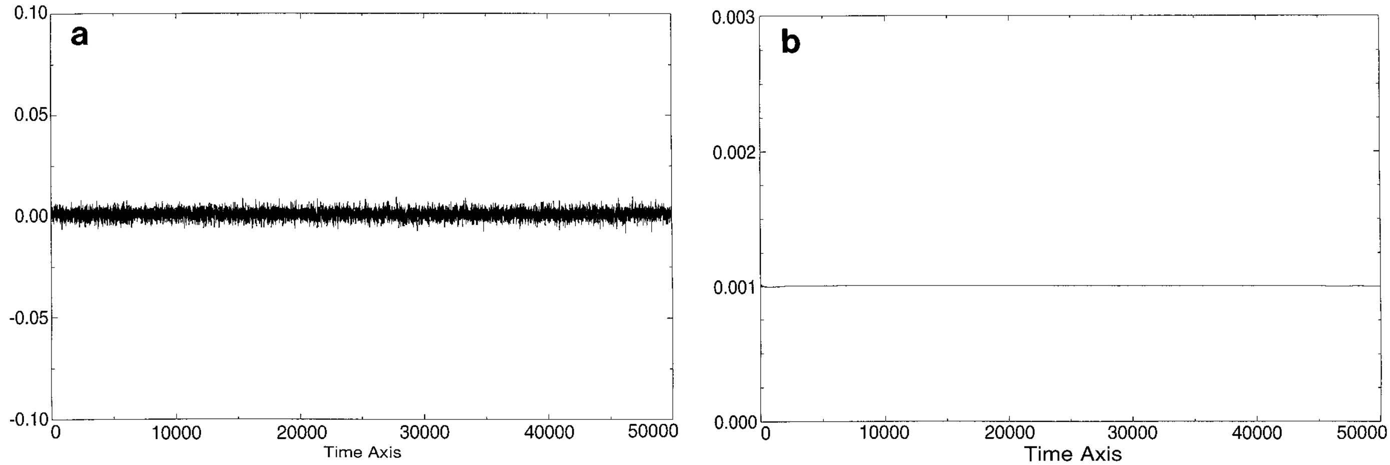

FIG. 2. (a) Instantaneous and (b) time-averaged values of $z(t)$.

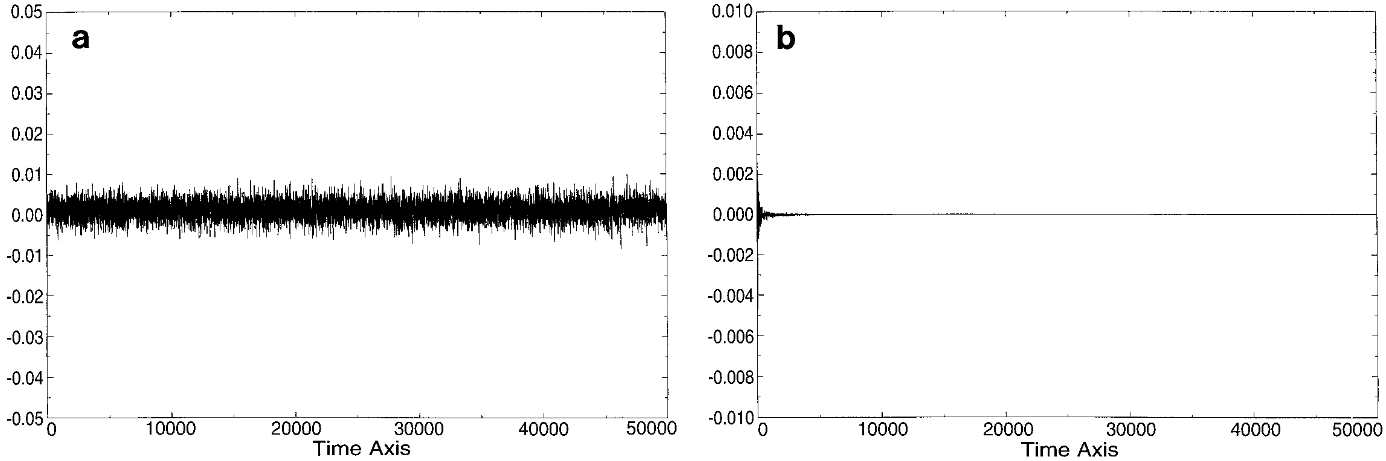

FIG. 3. (a) Instantaneous and (b) time-averaged values of $dy(t)/dt$.

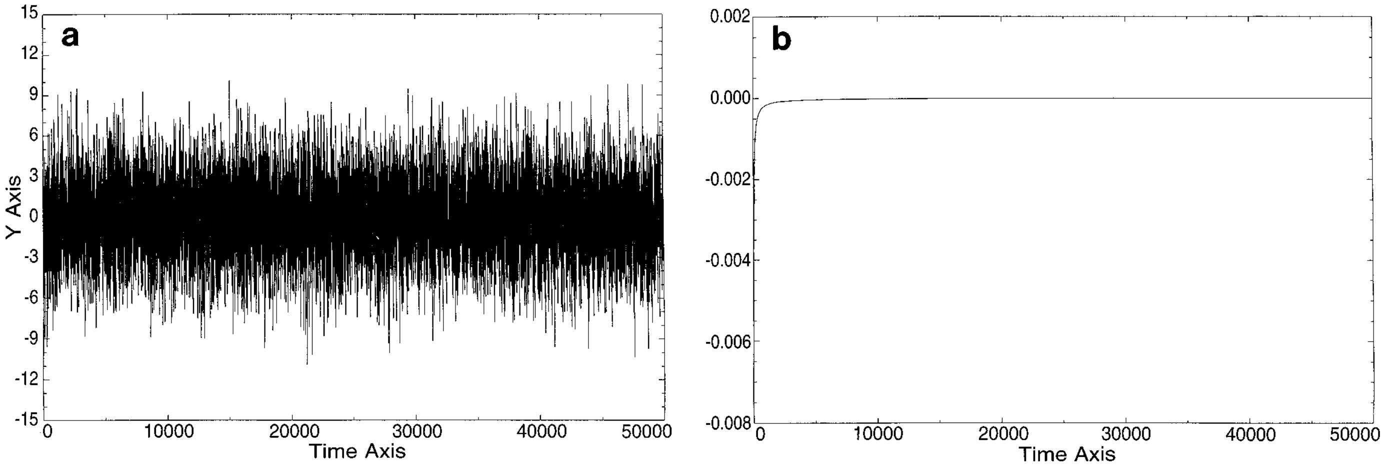

FIG. 4. (a) Instantaneous and (b) time-averaged values of $dz(t)/dt$.

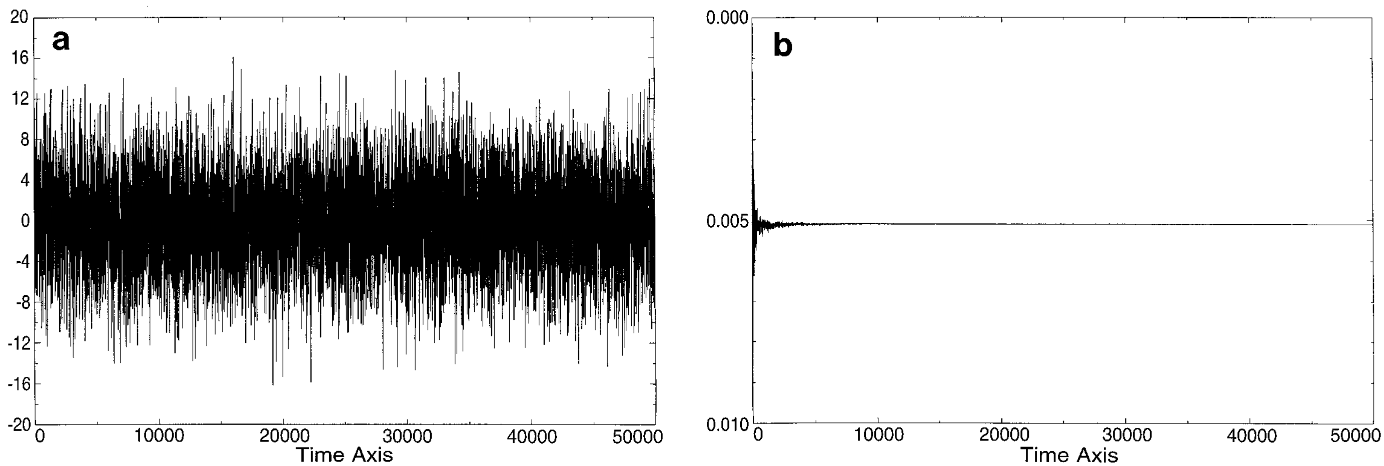

FIG. 5. (a) Instantaneous and (b) time-averaged values of $(dy(t)/dt)/y(t)$.

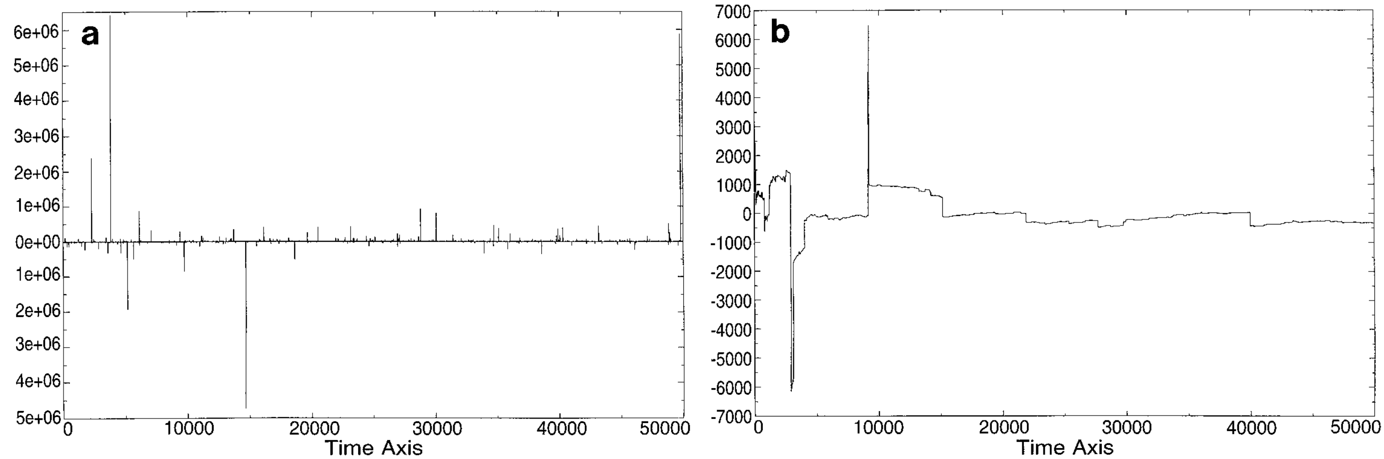

FIG. 6. (a) Instantaneous and (b) time-averaged values of $(dz(t)/dt)/z(t)$.

CONCLUDING REMARKS

We have presented a very simple model which is intended to mimic some of the features of a two-level discretization of the Navier–Stokes equations. By analogy with the incremental unknowns (IU) or nonlinear Galerkin method (NLG), we have shown that much computing time can be saved by using an explicit scheme for the low frequency/coarse grid part of the solution and an implicit scheme for the high frequency/fine grid part of the solution. With a mesh Δt for y and $q\ \Delta t$ for z we can essentially, as done in [10], freeze the high frequencies during the integration of the low frequencies. Another feature of the IU (or NLG) method which is not present here, is the utilization of a *nonlinear filter* $z = \Phi(y)$ provided by an approximate inertial manifold which can supplement or replace the integration of the z equation ($z^{n+1} = \Phi(y^{n+1})$, instead of (15); see [10] for the details).

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